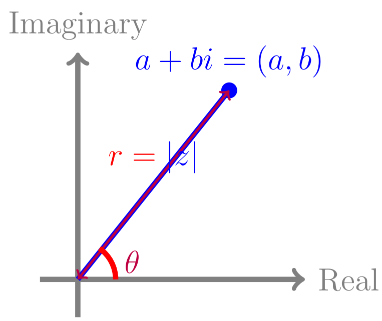
8.5: Polar Form of Complex Numbers

- Expressing $z = a + bi$ in the Polar Coordinate:



- Polar Form of a Complex Number

Writing a complex number, $a + bi$, in polar form, involves the following conversion formulas:

$a = |z|\cos(\theta)$, $b = |z|\sin(\theta)$.

Where absolute value of z is

$|z| = r = \sqrt{a^2 + b^2}$ is a real number called absolute value of z and θ is the angle made with polar axis in the polar form.

By direct substitution:

$$z = a + bi = (|z|\cos(\theta) + i|z|\sin(\theta)) = |z|\Big(\cos(\theta) + i\sin(\theta)\Big)$$

$|z|$ is called **Modulus** and θ is called **argument** of z. In your book, we abbreviate $z = |z|cis(\theta)$. This is also called Euler Identity and is represented with $z = |z|e^{i\theta}$

Complex Operations Using the Polar Form:

- Multiplication:

If $z_1 = |z_1|\Big(\cos(\theta_1) + i\sin(\theta_1)\Big)$ and $z_2 = |z_2|\Big(\cos(\theta_2) + i\sin(\theta_2)\Big)$, then

$$z_1.z_2 = |z_1||z_2|\Big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\Big)$$

Notice that the product calls for multiplying the moduli and adding the angles.

- Division:

If $z_1 = |z_1|\Big(\cos(\theta_1) + i\sin(\theta_1)\Big)$ and $z_2 = |z_2|\Big(\cos(\theta_2) + i\sin(\theta_2)\Big)$, then

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\Big(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\Big) \text{ where } z_2 \neq 0.$$

Notice that in finding the quotient of two complex numbers, you divide the moduli and subtract the arguments.

- De Moivres Theorem. (Finding an Integer Power of a Complex Number

  If $z = |z|\Big(\cos(\theta) + i\sin(\theta)\Big)$ is a complex number, then

  $z^n = |z|^n\Big(\cos(n\theta) + i\sin(n\theta)\Big)$ where n is a positive integer.

- Finding nth root of a Complex Number

  If $z = |z|\Big(\cos(\theta) + i\sin(\theta)\Big)$ is a complex number, then

  $z^{\frac{1}{n}} = |z|^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$ for $k = 0, 1, 2, ..., n-1$.

- Note on Calculations

  To be able to use any of the methods in operation with complex numbers, first find the polar coordinates of those complex numbers.

1. Express the following complex numbers in the polar form.

   (a) $5 + 5i$

   (b) $3 - 3i$

   (c) $1 + \sqrt{3}i$

   (d) $-1 + \sqrt{3}i$

Solution:

Use $|z| = \sqrt{a^2 + b^2}$ *and* $\tan(\theta) = \frac{b}{a}$ *in each case. Then plot to see if you are correct.*

(a) $5 + 5i = 5\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = \boxed{5\sqrt{2}e^{i\frac{\pi}{4}}}$

(b) $3 - 3i = 3\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \boxed{3\sqrt{2}e^{-i\frac{\pi}{4}}}$

(c) $1 + \sqrt{3}i = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) = \boxed{2e^{i\frac{\pi}{3}}}$

(d) $-1 + \sqrt{3}i = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = \boxed{2e^{i\frac{2\pi}{3}}}$

2. Perform the following operations.

(a) $\left[5\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)\right].\left[\sqrt{2}\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)\right]$ (a.k.a. $5e^{\frac{\pi}{3}i}.\sqrt{2}e^{\frac{\pi}{6}i}$)

(b) $\left[5\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)\right].\left[\sqrt{2}\left(\cos\left(\frac{\pi}{6}\right)-i\sin\left(\frac{\pi}{6}\right)\right)\right]$ (a.k.a. $5e^{\frac{2\pi}{3}i}.\sqrt{2}e^{\frac{-\pi}{6}i}$)

(c) $\left[5\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)\right].\left[-\sqrt{2}\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)\right]$

(d) $\left[5\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)\right].\left[\sqrt{2}\left(-\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)\right]$

(e) $\dfrac{\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)}{\sqrt{2}\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)}$ (a.k.a. $\frac{e^{\frac{2\pi}{3}i}}{\sqrt{2}e^{\frac{\pi}{6}i}}$)

(f) $\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)^7$ (a.k.a. $\left(e^{\frac{2\pi}{3}i}\right)^7$)

Solution:

(a) $\left[5\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)\right].\left[\sqrt{2}\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)\right]=$

$= 5e^{i\pi/3}.\sqrt{2}e^{i\pi/6} = 5\sqrt{2}e^{i\pi/2}$

$= \boxed{5\sqrt{2}\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)} = 5\sqrt{2}i$

(b) $\left[5\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)\right].\left[\sqrt{2}\left(\cos\left(\frac{\pi}{6}\right)-i\sin\left(\frac{\pi}{6}\right)\right)\right]$

$= \left[5\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)\right].\left[\sqrt{2}\left(\cos\left(\frac{-\pi}{6}\right)+i\sin\left(\frac{-\pi}{6}\right)\right)\right]=$

$= 5e^{i2\pi/3}.\sqrt{2}e^{-i\pi/6} = 5\sqrt{2}e^{i\pi/2}$

$= \boxed{5\sqrt{2}\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)} = 5\sqrt{2}i$

(c) $\left[5\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)\right].\left[-\sqrt{2}\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)\right]$

$= \left[5\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)\right].\left[\sqrt{2}\left(\cos\left(\frac{7\pi}{6}\right)+i\sin\left(\frac{7\pi}{6}\right)\right)\right]$

$= 5e^{i2\pi/3}.\sqrt{2}e^{i7\pi/6} = 5\sqrt{2}e^{i11\pi/6}$

$= \boxed{5\sqrt{2}\left(\cos\left(\frac{11\pi}{6}\right)+i\sin\left(\frac{11\pi}{6}\right)\right)} = 5\sqrt{2}\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)$

(d) $\left[5\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)\right].\left[-\sqrt{2}\left(-\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)\right]$

$= \left[5\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)\right].\left[\sqrt{2}\left(\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right)\right]$

$= 5e^{i2\pi/3}.\sqrt{2}e^{5i\pi/6} = 5\sqrt{2}e^{i3\pi/2}$

$= \boxed{5\sqrt{2}\left(\cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right)\right)} = -5\sqrt{2}i$

(e) $$\frac{\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)}{\sqrt{2}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)}$$

$$= \frac{e^{i2\pi/3}}{\sqrt{2}e^{i5\pi/6}} = \frac{1}{\sqrt{2}}e^{i\pi/2}$$

$$= \boxed{\frac{1}{\sqrt{2}}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)} = \frac{1}{\sqrt{2}}i$$

(f) $$\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)^7$$

$$= \left(e^{i2\pi/3}\right)^7 = e^{i4\pi/3}$$

$$= \boxed{\left(\cos\left(\frac{14\pi}{3}\right) + i\sin\left(\frac{14\pi}{3}\right)\right)} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

3. Perform the following operations.

(a) $\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)^{\frac{1}{2}}$

(b) $\left(128\left(\cos\left(\frac{7\pi}{3}\right)+i\sin\left(\frac{7\pi}{3}\right)\right)\right)^{\frac{1}{7}}$

Solution:

(a) $\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)^{\frac{1}{2}} = \boxed{\left(\cos\left(\frac{\pi}{3}+k\pi\right)+i\sin\left(\frac{\pi}{3}+k\pi\right)\right)}$ *for* $k = 0, 1$ *so* $\boxed{=\frac{1}{2}+i\frac{\sqrt{3}}{2}}$ *or* $\boxed{-\frac{1}{2}-i\frac{\sqrt{3}}{2}}$

(b) $\left(128\left(\cos\left(\frac{7\pi}{3}\right)+i\sin\left(\frac{7\pi}{3}\right)\right)\right)^{\frac{1}{7}} = 2\left(\cos\left(\frac{\pi}{3}+\frac{2k\pi}{7}\right)+i\sin\left(\frac{\pi}{3}+\frac{2k\pi}{7}\right)\right)$ *for* $k = 0, 1, 2, 3, 4, 5, 6.$

4. Find all complex solutions to the following equations.

(a) $x^3 = 8$

(b) $(x-1)^3 = 8$

Solution:

(a) $8 = 8(\cos(0)+i\sin(0)) \implies x = 8^{\frac{1}{3}} = 2(\cos\left(\frac{2k\pi}{3}\right)+i\sin\left(\frac{2k\pi}{3}\right))$ *for* $k = 0, 1, 2 \implies \boxed{x = 2, -1+\sqrt{3}i, -1-\sqrt{3}i}$

(b) *U-substitution* $u = x-1 \implies u = 2, -1+\sqrt{3}i, -1-\sqrt{3}i \implies \boxed{x = 3, \sqrt{3}i, -\sqrt{3}i}$